Approximation Rates for Deep ReLU Neural Networks on Sobolev Spaces

Deep neural networks, and especially deep ReLU neural networks, have recently driven significant breakthroughs in a variety of fields including scientific computing. In particular, the solution of partial differential equations (PDEs) using deep neural networks has been a major area of focus. One important component in understanding how neural networks solve PDEs is the approximation theory governing how efficiently they can approximate the PDE solution. Optimal $L^\infty$-approximation rates for deep ReLU networks on Lipschitz and $C^k$ spaces have recently been determined in the literature. In this work, we determine optimal $L^q$-approximation rates for deep ReLU networks on Sobolev spaces $W^s(L\_p)$, for which the smoothness is measured in the weaker $L\_p$-norm instead of in $L\_\infty$. The most interesting case is when $q > p$, i.e. when the error norm is stronger than the norm on the derivatives, in which case non-linear methods are particularly useful. Finally, we discuss implications for the numerical solution of PDEs using neural networks.