Modern Coarse Spaces in Domain Decomposition I

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Abstract

In 1987, Nicolaides proposed a deflation technique for the Conjugate Gradient method based on piecewise constant functions to improve its convergence [1]. According to Nicolaides, the technique can be used alone, or together with preconditioning, and Nicolaides mentions explicitly a relation to multigrid methods in the latter case. In the same year, Dryja and Widlund invented the additive variant of the overlapping Schwarz alternating method [2], where they used triangular subdomains, and piecewise linear functions defined on these subdomains as coarse space. Mandel and Brezina then used the Nicolaides coarse space for balancing domain decomposition in the non-overlapping Neumann-Neumann method [3], and Mandel and Tezaur introduced a similar technique in 1996 for FETI methods [4].

All these two level methods have optimal convergence properties when used as preconditioners for Krylov methods, see e.g the comprehensive book by Toselli and Widlund [5], where also optimal convergence for very general coarse spaces based on triangulations not aligned with the subdomains is proved. Optimal here means scalable, and not that these methods can not be further improved.

Over the last decade, spectral coarse spaces have become a new area of intensive research in domain decomposition. The main reason for this is that the classical optimal condition number estimates contain constants which depend on the coefficients in the partial differential equation that is solved, and these constants are not robust when the coefficients represent high contrast problems. This fact led researchers in domain decomposition to search for more effective coarse spaces for such problems, and it turned out that spectral information from the partial differential operator can be used effectively to this end. After early attempts using volume spectral information, it was realized that for domain decomposition spectral information of interface operators is more effective, and Dirichlet to Neumann operators were used to obtain spectral information [6]. This led, using the abstract Schwarz framework and its condition number estimate, to the very

successful GenEO coarse space [7] that can be generated algebraically and which directly improves the condition number estimate of the method.

While early on the emphasis was on high contrast problems, such coarse spaces can also greatly enhance the convergence of already optimal domain decomposition methods, see for example the Spectral Harmonically Enriched Multiscale Coarse Space (SHEM), which is based on the idea of an optimal coarse space in the sense of better is not possible [8], and is very much related to the fundamental idea of reduction in algebraic multigrid methods. I will show in my presentation visually how these new coarse space components look like for various domain decomposition methods for the simple Laplace problem, and show a first attempt to characterize them directly by closed form expressions using non-harmonic Fourier analysis [9].

References

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